

# On a Cahn–Hilliard–Darcy system for tumour growth with solution dependent source terms

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## Abstract

We study the existence of weak solutions to a mixture model for tumour growth that consists of a Cahn–Hilliard–Darcy system coupled with an elliptic reaction-diffusion equation. The Darcy law gives rise to an elliptic equation for the pressure that is coupled to the convective Cahn–Hilliard equation through convective and source terms. Both Dirichlet and Robin boundary conditions are considered for the pressure variable, which allows for the source terms to be dependent on the solution variables.

**Key words.** Cahn–Hilliard–Darcy system; phase field model; reaction-diffusion equation; tumour growth; chemotaxis; weak solutions; elliptic-parabolic system.

**AMS subject classification.** 35D30, 35Q35, 35Q92, 35K57, 76S05, 92C17, 92B05.

## 1 Introduction

At the fundamental level, cancer involves the unregulated growth of tissue inside the human body, which are caused by many biological and chemical mechanisms that take place at multiple spatial and temporal scales. In order to understand how these multiscale mechanisms are driving the progression of the cancer cells, whose dynamics may be too complex to be approached by experimental techniques, mathematical modelling can be used to provide a tractable description of the dynamics that isolate the key mechanisms and guide specific experiments.

We focus on the subclass of models for tumour growth known as diffuse interface models. These are continuum models that capture the macroscopic dynamics of the morphological changes of the tumour. For the simplest situation where there are only tumour cells and host cells in the presence of a nutrient, the model equations consists of a Cahn–Hilliard equation coupled to a reaction-diffusion equation for the nutrient. By treating the tumour and host cells as inertial-less fluids, a Darcy system can be appended to the Cahn–Hilliard equation, leading to a Cahn–Hilliard–Darcy system. For details regarding the diffuse interface models for tumour growth we refer the reader to [3, 6, 7, 16, 18, 21] and the references therein.

Our interest lies in providing analytical results for these models, namely in establishing the existence of solution to the model equations. Below, we introduce the Cahn–Hilliard–Darcy model to be studied: Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be a bounded domain with boundary

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$\Gamma$ , and denote, for  $T > 0$ ,  $Q := \Omega \times (0, T)$  and  $\Sigma := \Gamma \times (0, T)$ . We study the following elliptic-parabolic system:

$$\operatorname{div} \mathbf{v} = \Gamma_{\mathbf{v}}(\varphi, \sigma) \quad \text{in } Q, \quad (1.1a)$$

$$\partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) = \operatorname{div}(m(\varphi) \nabla \mu) + \Gamma_{\varphi}(\varphi, \sigma) \quad \text{in } Q, \quad (1.1b)$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi - \chi\sigma \quad \text{in } Q, \quad (1.1c)$$

$$0 = \Delta\sigma - h(\varphi)\sigma \quad \text{in } Q, \quad (1.1d)$$

$$\partial_{\nu}\varphi = 0, \quad \sigma = 1 \quad \text{on } \Sigma, \quad (1.1e)$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega, \quad (1.1f)$$

where  $\partial_{\nu}f := \nabla f \cdot \boldsymbol{\nu}$  is the normal derivative of  $f$  on the boundary  $\Gamma$ , with unit normal  $\boldsymbol{\nu}$ , and in this work, we focus on the following variants of Darcy's law and the boundary conditions

$$\mathbf{v} = -K(\nabla q + \varphi \nabla(\mu + \chi\sigma)) \quad \text{in } Q, \quad q = 0, \quad m(\varphi)\partial_{\nu}\mu = \varphi \mathbf{v} \cdot \boldsymbol{\nu} \quad \text{on } \Sigma, \quad (1.2a)$$

$$\mathbf{v} = -K(\nabla p - (\mu + \chi\sigma)\nabla\varphi) \quad \text{in } Q, \quad \mu = 0, \quad K\partial_{\nu}p = a(g - p) \quad \text{on } \Sigma, \quad (1.2b)$$

$$\mathbf{v} = -K(\nabla p - (\mu + \chi\sigma)\nabla\varphi) \quad \text{in } Q, \quad \partial_{\nu}\mu = 0, \quad K\partial_{\nu}p = a(g - p) \quad \text{on } \Sigma, \quad (1.2c)$$

for some positive constant  $a$  and prescribed function  $g$ . In (1.1),  $\mathbf{v}$  denotes the volume-averaged velocity of the cell mixture,  $\sigma$  denotes the concentration of the nutrient,  $\varphi \in [-1, 1]$  denotes the difference in volume fractions, with  $\{\varphi = 1\}$  representing unmixed tumour tissue, and  $\{\varphi = -1\}$  representing the host tissue, and  $\mu$  denotes the chemical potential for  $\varphi$ .

The positive constant  $K$  is the permeability of the mixture,  $m(\varphi)$  is a positive mobility for  $\varphi$ . The parameter  $\chi \geq 0$  regulates the chemotaxis effect (see [16] for more details),  $\Psi(\cdot)$  is a potential with two equal minima at  $\pm 1$ ,  $A$  and  $B$  denote two positive constants related to the thickness of the diffuse interface and the surface tension,  $h(\varphi)$  is an interpolation function that satisfies  $h(-1) = 0$  and  $h(1) = 1$ .

In (1.2), both  $p$  and  $q$  denote the pressure. The Darcy law in (1.2a) with pressure  $q$  can be obtained from the Darcy law in (1.2b) and (1.2c) with pressure  $p$  by setting  $q = p - (\mu + \chi\sigma)\varphi$ . The source terms  $\Gamma_{\mathbf{v}}$  and  $\Gamma_{\varphi}$  model, for instance, the growth of the tumour and its effect on the velocity field. We refer to [16, §2.5] for a discussion regarding the choices for the source terms  $\Gamma_{\varphi}, \Gamma_{\mathbf{v}}$ .

We now compare the model (1.1) with other models studied in the literature.

1. In the absence of velocity, i.e., setting  $\mathbf{v} = 0$  in (1.1b) and neglecting (1.1a), we obtain an elliptic-parabolic system that couples a Cahn–Hilliard equation with source term and an elliptic equation for the nutrient. A similar system has been studied by the authors in [12] with Dirichlet boundary conditions for  $\varphi, \mu, \sigma$ . For systems where (1.1d) has an additional  $\partial_t \sigma$  on the left-hand side, the well-posedness of solutions have been studied in [5, 11, 14] for particular choices of the source term  $\Gamma_{\varphi}$ . We also mention the work of [8] for the analysis of a system of equations similar to (1.1) with  $\chi = 0$ .
2. In the case  $\sigma = 0$ , (1.1) with the Darcy law (1.2b) reduces to a Cahn–Hilliard–Darcy system, and well-posedness results have been established in [20] for  $\Gamma_{\mathbf{v}} = \Gamma_{\varphi} = 0$  and  $\partial_{\nu}p = \partial_{\nu}\mu = 0$  on  $\Sigma$ , and in [19] for prescribed source terms  $\Gamma_{\mathbf{v}} = \Gamma_{\varphi} \neq 0$  and  $\partial_{\nu}p = \partial_{\nu}\mu = 0$  on  $\Sigma$ . In [2] a related system, known as the Cahn–Hilliard–Brinkman system, is studied, which features an additional term  $-\nu\Delta\mathbf{v}$  on the left-hand side of the Darcy law (1.2b), but with  $\Gamma_{\mathbf{v}} = \Gamma_{\varphi} = 0$ . Analogously, (1.1) without  $\sigma$  and

the Darcy law (1.2a) with boundary conditions  $\partial_\nu p = \partial_\nu \mu = \partial_\nu \varphi = 0$  on  $\Sigma$  has been studied in [10]. For strong solutions to the Cahn–Hilliard–Darcy system on the  $d$ -dimensional torus,  $d = 2, 3$ , we refer the reader to [23, 24].

3. In [13], the authors established the global existence of weak solutions to (1.1) with the Darcy law (1.2b) that features the following convection-reaction-diffusion equation for  $\sigma$ :

$$\partial_t \sigma + \operatorname{div}(\sigma \mathbf{v}) = \Delta \sigma - \chi \Delta \varphi - \mathcal{S},$$

with a prescribed source term  $\Gamma_{\mathbf{v}}$  and source terms  $\Gamma_\varphi, \mathcal{S}$  that depend on  $\varphi, \sigma$  and  $\mu$  that have at most linear growth, along with the boundary conditions  $\partial_\nu \mu = \partial_\nu \varphi = \partial_\nu p = 0$  and a Robin boundary condition for  $\sigma$ .

For the analyses performed on Cahn–Hilliard–Darcy systems in the literature, many have consider Neumann boundary conditions. However, a feature of the Neumann conditions for  $p$  and  $\varphi$  is that

$$\int_{\Omega} \Gamma_{\mathbf{v}} \, dx = \int_{\Omega} \operatorname{div} \mathbf{v} \, dx = \int_{\Gamma} \mathbf{v} \cdot \boldsymbol{\nu} \, d\Gamma = \int_{\Gamma} -K \partial_\nu p + K(\mu + \chi \sigma) \partial_\nu \varphi \, d\Gamma = 0,$$

that is, the source term  $\Gamma_{\mathbf{v}}$  necessarily have zero mean. For source terms  $\Gamma_{\mathbf{v}}$  that depends on  $\varphi$  and  $\sigma$ , this property may not be satisfied in general. To allow for a source term that need not have zero mean, one method is to prescribe alternate boundary conditions for the pressure, see for example [4, §2.2.9] and [16, §2.4.4].

In this work, for the pressure, we consider analysing the model with a Dirichlet boundary condition and also a Robin boundary condition for the pressure. Then, the source term  $\Gamma_{\mathbf{v}}$  does not need to fulfil the zero mean condition. However, it turns out that in the derivation of a priori estimates for the model, we encounter the following:

- For the natural boundary condition  $\partial_\nu \mu = 0$  and the Robin boundary condition  $K \partial_\nu p = a(g - p)$  on  $\Sigma$ , we have to restrict our analysis to potentials  $\Psi$  that has quadratic growth (Theorem 2.3).
- To consider potentials with polynomial growth of order larger than two, we need to prescribe the boundary conditions (1.2a) and (1.2b) for the chemical potential  $\mu$  (Theorems 2.1 and 2.2).

Let us briefly motivate the choices in (1.2a) and (1.2b). Due to the quasi-static nature of the nutrient equation (1.1d), we do not obtain a natural energy identity for the system (1.1) in contrast to the models studied in [13, 14, 16]. For simplicity, let  $m(\varphi) = 1$ ,  $K = 1$  and consider testing (1.1b) with  $\mu + \chi \sigma$ , (1.1c) with  $\partial_t \varphi$ , the Darcy law (1.2b) with  $\mathbf{v}$ . Integrating by parts and upon adding leads to

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} A\Psi(\varphi) + \frac{B}{2} |\nabla \varphi|^2 \, dx + \int_{\Omega} |\nabla \mu|^2 + |\mathbf{v}|^2 \, dx \\ &= \int_{\Omega} -\chi \nabla \mu \cdot \nabla \sigma + \Gamma_{\mathbf{v}}(p - \varphi(\mu + \chi \sigma)) + \Gamma_{\varphi}(\mu + \chi \sigma) \, dx \\ &+ \int_{\Gamma} \partial_\nu \mu(\mu + \chi \sigma) - p \mathbf{v} \cdot \boldsymbol{\nu} \, d\Gamma. \end{aligned} \tag{1.3}$$

If we prescribe the boundary conditions  $\partial_\nu \mu = 0$  and  $-\mathbf{v} \cdot \boldsymbol{\nu} = \partial_\nu p = a(g - p)$ , i.e., the boundary conditions in (1.2c), then the boundary term in (1.3) poses no difficulties. The

main difficulty in obtaining a priori estimates from (1.3) is to control the source terms  $\Gamma_{\mathbf{v}}\mu\varphi$  and  $\Gamma_{\varphi}\mu$  with the left-hand side of (1.3). In the absence of any previous a priori estimates, to control terms involving  $\mu$  by the term  $\|\nabla\mu\|_{L^2(\Omega)}^2$  on the left-hand side via the Poincaré inequality, an estimate of the square of the mean of  $\mu$  is needed. As observed in [14], this leads to a restriction to quadratic growth assumptions for the potential  $\Psi$ .

Furthermore, new difficulties arise in estimating the source term  $\Gamma_{\mathbf{v}}p$  if we do not prescribe a Neumann boundary condition for  $p$ . The methodology used in [13, 19] to obtain an estimate for  $\|p\|_{L^2(\Omega)}$  relies on the assumption that  $\Gamma_{\mathbf{v}}$  is prescribed and has zero mean, and  $\partial_{\nu}p = 0$  on  $\Sigma$ . The arguments in [13, 19] seem not to be applicable for the our present setting, where  $\Gamma_{\mathbf{v}}$  is dependent on  $\varphi$  and  $\sigma$ , and a Robin boundary condition is prescribed for  $p$ . This motivates the choice of a Dirichlet condition for  $\mu$  to handle the source term  $\Gamma_{\mathbf{v}}\varphi\mu$  and  $\Gamma_{\varphi}\mu$ , and as we will see later in Section 3.4 (specifically (3.20)), the Dirichlet boundary condition for  $\mu$  is needed to obtain an  $L^2$ -estimate for  $p$ .

Alternatively, we may consider the discussion in [13, §8] regarding reformulations of the Darcy law. Choosing  $q = p - \varphi(\mu + \chi\sigma)$  leads to the Darcy law variant in (1.2a). A similar testing procedure leads to

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} A\Psi(\varphi) + \frac{B}{2} |\nabla\varphi|^2 \, dx + \int_{\Omega} |\nabla\mu|^2 + |\mathbf{v}|^2 \, dx \\ &= \int_{\Omega} -\chi \nabla\mu \cdot \nabla\sigma + \Gamma_{\mathbf{v}}q + \Gamma_{\varphi}(\mu + \chi\sigma) \, dx \\ &+ \int_{\Gamma} (\partial_{\nu}\mu - \varphi\mathbf{v} \cdot \boldsymbol{\nu})(\mu + \chi\sigma) - q\mathbf{v} \cdot \boldsymbol{\nu} \, d\Gamma. \end{aligned} \tag{1.4}$$

Here we observed that the source term involving  $\Gamma_{\mathbf{v}}$  simplifies to just  $\Gamma_{\mathbf{v}}q$ , and in exchange, we see the appearance of  $(q + \varphi\mu + \chi\varphi\sigma)\mathbf{v} \cdot \boldsymbol{\nu}$  appearing in the boundary term. Comparing to the previous set-up with (1.2b), we have shifted the problematic terms to the boundary integral. Choosing  $\mathbf{v} \cdot \boldsymbol{\nu} = 0$  on  $\Sigma$  is not desirable, as equation (1.1a) would imply that  $\Gamma_{\mathbf{v}}(\varphi, \sigma)$  must have zero mean. We may instead consider the boundary conditions

$$\partial_{\nu}\mu = 0, \quad \mathbf{v} \cdot \boldsymbol{\nu} = -\partial_{\nu}(q + \chi\sigma) = a(q + \varphi(\mu + \chi\sigma)) \text{ on } \Sigma,$$

then the boundary term in (1.3) poses no additional difficulties in obtaining a priori estimate. In exchange, obtaining an estimate for  $\|q\|_{L^2(\Omega)}$  to deal with the source term  $\Gamma_{\mathbf{v}}q$  becomes more involved, as the variational formulation for the pressure system now reads as

$$\int_{\Omega} \nabla q \cdot \nabla \zeta \, dx + \int_{\Gamma} a q \zeta \, d\Gamma = \int_{\Omega} \Gamma_{\mathbf{v}}\zeta - \varphi \nabla(\mu + \chi\sigma) \cdot \nabla \zeta \, dx - \int_{\Gamma} a \varphi(\mu + \chi\sigma) \zeta \, d\Gamma$$

for a test function  $\zeta$ . Estimates for  $q$  will now involve an estimate for  $\|\varphi\mu\|_{L^2(\Gamma)}$ , and this is more difficult to control than  $\|\varphi\mu\|_{L^2(\Omega)}$ . This motivates the choice of a Dirichlet condition for  $q$  and the boundary condition  $\partial_{\nu}\mu = \varphi\mathbf{v} \cdot \boldsymbol{\nu}$  to eliminate the boundary term in (1.4).

This paper is organised as follows. In Section 2 we state the main assumptions and the main results. In Section 3 we outline the proof of the existence result by means of a Schauder fixed point argument. An auxiliary problem involving just the nutrient is studied in Section 3.1, and an auxiliary problem involving Cahn–Hilliard–Darcy system with (1.2a) is studied in Section 3.2. Then, in Section 3.3 the Schauder’s fixed point theorem is applied to deduce the existence of a weak solution to (1.1), (1.2a). The details for the Robin boundary conditions (1.2b) and (1.2c) are specified in Sections 3.4 and 3.5, respectively.

**Notation.** For convenience, we will often use the notation  $L^p := L^p(\Omega)$  and  $W^{k,p} := W^{k,p}(\Omega)$  for any  $p \in [1, \infty]$ ,  $k > 0$  to denote the standard Lebesgue spaces and Sobolev spaces equipped with the norms  $\|\cdot\|_{L^p}$  and  $\|\cdot\|_{W^{k,p}}$ . In the case  $p = 2$  we use  $H^k := W^{k,2}$  and the norm  $\|\cdot\|_{H^k}$ . Due to the Dirichlet boundary condition for  $\sigma$  and  $\mu$ , we denote the space  $H_0^1$  as the completion of  $C_c^\infty(\Omega)$  with respect to the  $H^1$  norm. We will use the isometric isomorphism  $L^p(Q) \cong L^p(0, T; L^p)$  and  $L^p(\Sigma) \cong L^p(0, T; L^p(\Gamma))$  for any  $p \in [1, \infty)$ . Moreover, the dual space of a Banach space  $X$  will be denoted by  $X^*$ , and the duality pairing between  $X$  and  $X^*$  is denoted by  $\langle \cdot, \cdot \rangle_X$ . We denote the dual space to  $H_0^1$  as  $H^{-1}$ . For  $d = 2$  or  $3$ , let  $d\Gamma$  denote integration with respect to the  $(d-1)$  dimensional Hausdorff measure on  $\Gamma$ , and we denote  $\mathbb{R}^d$ -valued functions in boldface. For convenience, we will often use the notation

$$\int_Q f := \int_0^T \int_\Omega f \, dx \, dt, \quad \int_{\Omega_t} f := \int_0^t \int_\Omega f \, dx \, ds, \quad \int_{\Gamma_t} f := \int_0^t \int_\Gamma f \, d\Gamma \, ds$$

for any  $f \in L^1(Q)$  and for any  $t \in (0, T]$ .

**Useful preliminaries.** For convenience, we recall the Poincaré inequality: There exist a positive constant  $C_p$  depending only on  $\Omega$  such that

$$\|f - \bar{f}\|_{L^r} \leq C_p \|\nabla f\|_{L^r} \text{ for all } f \in W^{1,r}, 1 \leq r \leq \infty, \quad (1.5)$$

where  $\bar{f} := \frac{1}{|\Omega|} \int_\Omega f \, dx$  denotes the mean of  $f$ . Furthermore, we have

$$\|f\|_{L^2} \leq C_p (\|\nabla f\|_{L^2} + \|f\|_{L^2(\Gamma)}) \text{ for } f \in H^1, \quad (1.6)$$

$$\|f\|_{L^2} \leq C_p \|\nabla f\|_{L^2} \text{ for } f \in H_0^1. \quad (1.7)$$

The Gagliardo–Nirenberg interpolation inequality in dimension  $d$  is also useful (see [9, Theorem 2.1] and [1, Theorem 5.8]): Let  $\Omega$  be a bounded domain with Lipschitz boundary, and  $f \in W^{m,r} \cap L^q$ ,  $1 \leq q, r \leq \infty$ . For any integer  $j$ ,  $0 \leq j < m$ , suppose there is  $\alpha \in \mathbb{R}$  such that

$$\frac{1}{p} = \frac{j}{d} + \left( \frac{1}{r} - \frac{m}{d} \right) \alpha + \frac{1-\alpha}{q}, \quad \frac{j}{m} \leq \alpha \leq 1.$$

There exists a positive constant  $C$  depending only on  $\Omega$ ,  $m$ ,  $j$ ,  $q$ ,  $r$ , and  $\alpha$  such that

$$\|D^j f\|_{L^p} \leq C \|f\|_{W^{m,r}}^\alpha \|f\|_{L^q}^{1-\alpha}. \quad (1.8)$$

For  $f \in L^2$ ,  $g \in L^2(\Gamma)$ , and  $\beta > 0$ , let  $u \in H^1$ ,  $w \in H_0^1$  be the unique solutions to the elliptic problems

$$\begin{aligned} -\Delta w &= f \text{ in } \Omega, & w &= 0 & \text{ on } \Gamma, \\ -\Delta u &= f \text{ in } \Omega, & \partial_\nu u + \beta u &= g & \text{ on } \Gamma. \end{aligned}$$

We use the notation  $u = (-\Delta_R)^{-1}(f, \beta, g)$  and  $w = (-\Delta_D)^{-1}(f)$ . Furthermore, if in addition  $g \in H^{\frac{1}{2}}(\Gamma)$  and  $\Gamma$  is a  $C^2$ -boundary, then by elliptic regularity theory [17, Thm. 2.4.2.6] and [17, Thm. 2.4.2.5], it holds that  $w \in H^2 \cap H_0^1$  and  $u \in H^2$  with

$$\|w\|_{H^2} \leq C \|f\|_{L^2}, \quad \|u\|_{H^2} \leq C \left( \|f\|_{L^2} + \|g\|_{H^{\frac{1}{2}}(\Gamma)} \right).$$

## 2 Assumptions and main results

### Assumption 2.1.

- (A1)  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , is a bounded domain with  $C^3$ -boundary  $\Gamma$ . The positive constants  $a, T, A, B, \chi, K$  are fixed. The function  $g \in L^2(\Sigma)$  and the initial condition  $\varphi_0 \in H^1$  are prescribed.
- (A2) The mobility  $m \in C^0(\mathbb{R})$  satisfies  $0 < m_0 \leq m(s) \leq m_1$  for all  $s \in \mathbb{R}$ . The function  $h \in C^0(\mathbb{R})$  is non-negative and is bounded above by 1.
- (A3) The potential  $\Psi \in C^2(\mathbb{R})$  is non-negative and, for  $r \in [0, 4)$  and for all  $s \in \mathbb{R}$ , there exist positive constants  $C_1, C_2, C_3$  and  $C_4$  such that

$$\Psi(s) \geq C_1 |s|^2 - C_2, \quad |\Psi''(s)| \leq C_3 (1 + |s|^r), \quad |\Psi'(s)| \leq C_4 (1 + \Psi(s)).$$

- (A4) The source terms  $\Gamma_{\mathbf{v}}$  and  $\Gamma_{\varphi}$  are of the form

$$\Gamma_{\mathbf{v}}(\varphi, \sigma) = b_{\mathbf{v}}(\varphi)\sigma + f_{\mathbf{v}}(\varphi), \quad \Gamma_{\varphi}(\varphi, \sigma) = b_{\varphi}(\varphi)\sigma + f_{\varphi}(\varphi),$$

where  $b_{\mathbf{v}}, b_{\varphi}, f_{\mathbf{v}}, f_{\varphi}$  are bounded and continuous functions.

We first give the results to the problem (1.1), (1.2a).

**Definition 2.1.** We call a quintuple  $(\varphi, \mu, \sigma, \mathbf{v}, q)$  a weak solution to (1.1), (1.2a) if

$$\begin{aligned} \varphi &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^3) \cap W^{1, \frac{8}{5}}(0, T; (H^1)^*), \quad \mathbf{v} \in L^2(Q), \\ \sigma &\in (1 + L^2(0, T; H_0^1)), \quad \mu \in L^2(0, T; H^1), \quad q \in L^{\frac{8}{5}}(0, T; H_0^1), \end{aligned}$$

and satisfies  $\varphi(0) = \varphi_0$ ,  $0 \leq \sigma \leq 1$  a.e. in  $Q$ , and

$$0 = \langle \partial_t \varphi, \zeta \rangle_{H^1} + \int_{\Omega} m(\varphi) \nabla \mu \cdot \nabla \zeta - \varphi \mathbf{v} \cdot \nabla \zeta - \Gamma_{\varphi}(\varphi, \sigma) \zeta \, dx, \quad (2.1a)$$

$$0 = \int_{\Omega} (\mu + \chi \sigma) \zeta - A \Psi'(\varphi) \zeta - B \nabla \varphi \cdot \nabla \zeta \, dx, \quad (2.1b)$$

$$0 = \int_{\Omega} \nabla \sigma \cdot \nabla \xi + h(\varphi) \sigma \xi \, dx, \quad (2.1c)$$

$$0 = \int_{\Omega} K \nabla q \cdot \nabla \xi - \Gamma_{\mathbf{v}}(\varphi, \sigma) \xi + K \varphi \nabla (\mu + \chi \sigma) \cdot \nabla \xi \, dx, \quad (2.1d)$$

$$0 = \int_{\Omega} \mathbf{v} \cdot \mathbf{y} + K \nabla q \cdot \mathbf{y} + K \varphi \nabla (\mu + \chi \sigma) \cdot \mathbf{y} \, dx, \quad (2.1e)$$

for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\xi \in H_0^1$ ,  $\mathbf{y} \in L^2$ .

**Theorem 2.1.** Under Assumption 2.1, there exists a weak solution to (1.1), (1.2a) in the sense of Definition 2.1.

For the problem (1.1), (1.2b) we have the following.

**Definition 2.2.** We call a quintuple  $(\varphi, \mu, \sigma, \mathbf{v}, p)$  a weak solution to (1.1), (1.2b) if

$$\begin{aligned} \varphi &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^3) \cap W^{1, \frac{8}{5}}(0, T; H^{-1}), \quad \mathbf{v} \in L^2(Q), \\ \sigma &\in (1 + L^2(0, T; H_0^1)), \quad \mu \in L^2(0, T; H_0^1), \quad p \in L^{\frac{8}{5}}(0, T; H^1) \cap L^2(\Sigma), \end{aligned}$$

and satisfies  $\varphi(0) = \varphi_0$ ,  $0 \leq \sigma \leq 1$  a.e. in  $Q$ , (2.1b), (2.1c), and

$$0 = \langle \partial_t \varphi, \xi \rangle_{H_0^1} + \int_{\Omega} m(\varphi) \nabla \mu \cdot \nabla \xi - \varphi \mathbf{v} \cdot \nabla \xi - \Gamma_{\mathbf{v}}(\varphi, \sigma) \xi \, dx, \quad (2.2a)$$

$$0 = \int_{\Omega} K \nabla p \cdot \nabla \zeta - \Gamma_{\mathbf{v}}(\varphi, \sigma) \zeta - K(\mu + \chi \sigma) \nabla \varphi \cdot \nabla \zeta \, dx + \int_{\Gamma} a(p - g) \zeta \, d\Gamma, \quad (2.2b)$$

$$0 = \int_{\Omega} \mathbf{v} \cdot \mathbf{y} + K \nabla p \cdot \mathbf{y} - K(\mu + \chi \sigma) \nabla \varphi \cdot \mathbf{y} \, dx, \quad (2.2c)$$

for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\xi \in H_0^1$ ,  $\mathbf{y} \in L^2$ .

**Theorem 2.2.** *Under Assumption 2.1, there exists a weak solution to (1.1), (1.2b) in the sense of Definition 2.2.*

Analogously for the problem (1.1), (1.2c) we have the following.

**Definition 2.3.** *We call a quintuple  $(\varphi, \mu, \sigma, \mathbf{v}, p)$  a weak solution to (1.1), (1.2c) if*

$$\begin{aligned} \varphi &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^3) \cap W^{1, \frac{8}{5}}(0, T; (H^1)^*), \quad \mathbf{v} \in L^2(Q), \\ \sigma &\in (1 + L^2(0, T; H_0^1)), \quad \mu \in L^2(0, T; H^1), \quad p \in L^{\frac{8}{5}}(0, T; H^1) \cap L^2(\Sigma), \end{aligned}$$

and satisfies  $\varphi(0) = \varphi_0$ ,  $0 \leq \sigma \leq 1$  a.e. in  $Q$ , (2.1b), (2.1c), (2.2b) and (2.2c) and

$$0 = \langle \partial_t \varphi, \zeta \rangle_{H^1} + \int_{\Omega} m(\varphi) \nabla \mu \cdot \nabla \zeta + \nabla \varphi \cdot \mathbf{v} \zeta + \Gamma_{\mathbf{v}}(\varphi, \sigma) \varphi \zeta - \Gamma_{\varphi}(\varphi, \sigma) \zeta \, dx, \quad (2.3)$$

for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\xi \in H_0^1$ ,  $\mathbf{y} \in L^2$ .

**Theorem 2.3.** *Under Assumption 2.1, with (A3) replaced by*

$$\Psi(s) \geq C_1 |s|^2 - C_2, \quad |\Psi''(s)| \leq C_3 \quad \forall s \in \mathbb{R}, \quad (2.4)$$

for positive constants  $C_1, C_2, C_3$ , there exists a weak solution to (1.1), (1.2c) in the sense of Definition 2.3.

We use the fact that  $H^1 \subset\subset L^2 \subset (H^1)^*$ ,  $H^1 \subset\subset L^2 \subset H^{-1}$ , and [22, §8, Cor. 4] to deduce that  $\varphi \in C^0([0, T]; L^2)$  in all cases, and thus  $\varphi(0)$  makes sense as a function in  $L^2$ . This implies that the initial condition  $\varphi_0$  is attained in all cases.

### 3 Existence

We show the existence of weak solutions to (1.1), (1.2a) by means of a fixed point argument. The idea, similarly applied in [15], is to consider the following two auxiliary problems. For a given  $\phi \in L^2(Q)$ , let  $\sigma$  be a solution to the auxiliary problem

$$-\Delta \sigma = h(\phi) \sigma \text{ in } Q, \quad \sigma = 1 \text{ on } \Sigma. \quad (3.1)$$

This defines a mapping  $\mathcal{L} : \phi \mapsto \sigma$ . Then, we find a quadruple  $(\varphi, \mu, \mathbf{v}, q)$  of functions that is a weak solution to the auxiliary problem

$$\operatorname{div} \mathbf{v} = \Gamma_{\mathbf{v}}(\varphi, \mathcal{L}(\phi)) \quad \text{in } Q, \quad (3.2a)$$

$$\mathbf{v} = -K(\nabla q + \varphi \nabla(\mu + \chi \mathcal{L}(\phi))) \quad \text{in } Q, \quad (3.2b)$$

$$\partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) = \operatorname{div}(m(\varphi) \nabla \mu) + \Gamma_{\varphi}(\varphi, \mathcal{L}(\phi)) \quad \text{in } Q, \quad (3.2c)$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi - \chi\mathcal{L}(\phi) \quad \text{in } Q, \quad (3.2d)$$

$$\partial_{\boldsymbol{\nu}}\varphi = 0, \quad m(\varphi)\partial_{\boldsymbol{\nu}}\mu = \varphi\mathbf{v} \cdot \boldsymbol{\nu}, \quad q = 0 \text{ on } \Sigma, \quad (3.2e)$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega. \quad (3.2f)$$

This yields a mapping  $\mathcal{M} : \phi \mapsto \varphi$ . If  $\varphi_*$  is a fixed point of  $\mathcal{M}$ , i.e.,  $\varphi_* = \mathcal{M}(\varphi_*)$ , with  $\sigma_* = \mathcal{L}(\varphi_*)$ ,  $\mu_* = A\Psi'(\varphi_*) - B\Delta\varphi_* - \chi\sigma_*$ ,  $\mathbf{v}_* = -K(\nabla q_* + \varphi_*\nabla(\mu_* + \chi\sigma_*))$  and  $\text{div } \mathbf{v}_* = \Gamma_{\mathbf{v}}(\varphi_*, \sigma_*)$ , then  $(\varphi_*, \mu_*, \sigma_*, \mathbf{v}_*, q_*)$  is a solution to (1.1), (1.2a). A similar strategy will also be used for showing the existence of weak solutions to (1.1), (1.2b) and (1.1), (1.2c).

### 3.1 Auxiliary nutrient equation

**Lemma 3.1.** *For any  $\phi \in L^2(Q)$ , there exists a solution  $\sigma \in (1 + L^2(0, T; H_0^1))$  to (3.1) such that  $0 \leq \sigma \leq 1$  a.e. in  $Q$ . Furthermore, there exists a constant  $C$  not depending on  $\phi$  such that  $\|\sigma\|_{L^2(0, T; H^1)} \leq C$ .*

*Proof.* Given  $\phi \in L^2(Q)$  and a function  $\sigma_0 \in L^2$ , for any  $\theta \in (0, 1]$ , we find a solution  $\sigma^{(\theta)}$  to the problem

$$\theta\partial_t\sigma - \Delta\sigma + h(\phi)\sigma = 0 \text{ in } Q, \quad \sigma = 1 \text{ on } \Sigma, \quad \sigma(0) = \sigma_0 \text{ in } \Omega. \quad (3.3)$$

Applying a standard Galerkin approximation yields the existence of a solution  $\sigma^{(\theta)}$  to (3.3) satisfying  $\sigma^{(\theta)} \in (1 + L^2(0, T; H_0^1)) \cap H^1(0, T; H^{-1})$ , where  $H^{-1}$  is the dual space to  $H_0^1$ . We now derive some uniform estimates for  $\sigma^{(\theta)}$ . Testing with  $\sigma^{(\theta)} - 1$  and integrating in time leads to

$$\begin{aligned} & \theta\|(\sigma^{(\theta)} - 1)(t)\|_{L^2}^2 + \|\nabla\sigma^{(\theta)}\|_{L^2(Q)}^2 + \int_Q h(\phi)(\sigma^{(\theta)} - 1)^2 \\ & \leq \theta\|\sigma_0 - 1\|_{L^2}^2 + \int_Q h(\phi)|\sigma^{(\theta)} - 1| \leq \|\sigma_0 - 1\|_{L^2}^2 + \frac{1}{2C_p}\|\sigma^{(\theta)} - 1\|_{L^2(Q)}^2 + C \\ & \leq \|\sigma_0 - 1\|_{L^2}^2 + \frac{1}{2}\|\nabla\sigma^{(\theta)}\|_{L^2(Q)}^2 + C, \end{aligned}$$

for all  $t \in (0, T]$ , where the constant  $C$  does not depend on  $\theta \in (0, 1]$  and  $C_p$  is the constant from the Poincaré inequality. Neglecting the non-negative terms  $\|(\sigma^{(\theta)} - 1)(t)\|_{L^2}^2$  and  $h(\phi)(\sigma^{(\theta)} - 1)^2$  in the above estimate leads to

$$\begin{aligned} \|\sigma^{(\theta)}\|_{L^2(0, T; H^1)} & \leq C \left(1 + \|(\sigma^{(\theta)} - 1)\|_{L^2(0, T; H^1)}\right) \\ & \leq C \left(1 + \|\nabla\sigma^{(\theta)}\|_{L^2(Q)}\right) \leq C. \end{aligned} \quad (3.4)$$

Furthermore, by the boundedness of  $h$  and the Poincaré inequality, it holds that

$$\begin{aligned} \|\theta\partial_t\sigma^{(\theta)}\|_{L^2(0, T; H^{-1})} & \leq \|\nabla\sigma^{(\theta)}\|_{L^2(Q)} + \|\sigma^{(\theta)} - 1\|_{L^2(Q)} \\ & \leq C \left(\|\nabla\sigma^{(\theta)}\|_{L^2(Q)} + 1\right), \end{aligned}$$

where  $C$  is a positive constant not dependent on  $\theta$ . These estimates show that

$$\begin{aligned} \{\sigma^{(\theta)}\}_{\theta \in (0, 1]} & \text{ is bounded in } 1 + L^2(0, T; H_0^1), \\ \{\theta\partial_t\sigma^{(\theta)}\}_{\theta \in (0, 1]} & \text{ is bounded in } L^2(0, T; H^{-1}), \end{aligned}$$



and thus there exists a function  $\sigma \in (1 + L^2(0, T; H_0^1))$  such that

$$\sigma^{(\theta)} \rightarrow \sigma \text{ weakly in } L^2(0, T; H^1).$$

Then, it is a standard argument to show that  $\sigma$  is a weak solution to (3.3) with  $\theta = 0$ . To deduce that the limit function  $\sigma$  satisfies  $0 \leq \sigma \leq 1$  a.e. in  $Q$ , we use a weak comparison principle. Testing (3.1) with  $(\sigma - 1)_+ := \max(\sigma - 1, 0)$ , we see that

$$\begin{aligned} \int_Q \nabla \sigma \cdot \nabla (\sigma - 1)_+ &= \|\nabla (\sigma - 1)_+\|_{L^2(Q)}^2 = - \int_Q h(\phi) \sigma (\sigma - 1)_+ \\ &= - \int_Q (h(\phi)(\sigma - 1)(\sigma - 1)_+ + h(\phi)(\sigma - 1)_+) \leq - \int_Q h(\phi) |(\sigma - 1)_+|^2 \leq 0. \end{aligned}$$

This shows that  $(\sigma - 1)_+$  is constant a.e. in  $Q$ , and  $(\sigma - 1)_+ = 0$  on  $\Sigma$  implies that  $(\sigma - 1)_+ = 0$  a.e. in  $Q$ , which yields that  $\sigma \leq 1$  a.e. in  $Q$ . Similarly, testing (3.1) with  $(\sigma)_- := \max(-\sigma, 0)$  leads to

$$-\|\nabla (\sigma)_-\|_{L^2(Q)}^2 = \int_Q \nabla \sigma \cdot \nabla (\sigma)_- = - \int_Q h(\phi) \sigma (\sigma)_- = \int_Q h(\phi) |(\sigma)_-|^2 \geq 0.$$

This shows that  $(\sigma)_-$  is constant a.e. in  $Q$ , and using that  $(\sigma)_- = 0$  on  $\Sigma$  leads to the assertion that  $(\sigma)_- = 0$  a.e. in  $Q$ , and so  $0 \leq \sigma$  a.e. in  $Q$ .  $\square$

### 3.2 Auxiliary Cahn–Hilliard–Darcy system

We state the existence result to (3.2):

**Lemma 3.2.** *Under Assumption 2.1, for any  $\phi \in L^2(Q)$  there exists a quadruple  $(\varphi, \mu, \mathbf{v}, q)$  such that*

$$\begin{aligned} \varphi &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^3) \cap W^{1, \frac{8}{5}}(0, T; (H^1)^*), \\ \mathbf{v} &\in L^2(Q), \quad q \in L^{\frac{8}{5}}(0, T; H_0^1), \quad \mu \in L^2(0, T; H^1), \end{aligned}$$

which satisfies (2.1a), (2.1b), (2.1d), (2.1e) (with  $\sigma$  replaced by  $\mathcal{L}(\phi)$ ) for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\mathbf{y} \in L^2$ ,  $\xi \in H_0^1$ . Furthermore, there exists a positive constant  $C$ , not depending on  $(\varphi, \mu, \mathbf{v}, q)$  and  $\phi$  such that

$$\begin{aligned} \|\Psi(\varphi)\|_{L^\infty(0, T; L^1)} + \|\Psi'(\varphi)\|_{L^2(0, T; H^1)} + \|\varphi\|_{L^\infty(0, T; H^1) \cap L^2(0, T; H^3)} \\ + \|\mu\|_{L^2(0, T; H^1)} + \|\mathbf{v}\|_{L^2(Q)} + \|q\|_{L^{\frac{8}{5}}(0, T; H_0^1)} + \|\partial_t \varphi\|_{L^{\frac{8}{5}}(0, T; (H^1)^*)} \leq C. \end{aligned} \quad (3.5)$$

To prove this result, we first derive a priori estimates for (3.2). In the following,  $C$  denotes a positive constant not depending on  $(\varphi, \mu, \mathbf{v}, q)$  or  $\phi$ , and may vary from line to line. We write  $\mathcal{L}(\phi)$  as  $\sigma$  in (3.2) and replace the duality product  $\langle \cdot, \cdot \rangle_{H^1}$  in (2.1a) with the  $L^2$ -product (this is satisfied for example by the Galerkin ansatz). Substituting  $\zeta = \partial_t \varphi$  in (2.1b),  $\zeta = \mu + \chi \sigma$  in (2.1a),  $\mathbf{y} = K^{-1} \mathbf{v}$  in (2.1e) and summing leads to

$$\begin{aligned} \frac{d}{dt} \int_\Omega A \Psi(\varphi) + \frac{B}{2} |\nabla \varphi|^2 \, dx + \int_\Omega m(\varphi) |\nabla \mu|^2 + \frac{1}{K} |\mathbf{v}|^2 \, dx \\ = \int_\Omega -m(\varphi) \chi \nabla \mu \cdot \nabla \sigma + \Gamma_\varphi(\mu + \chi \sigma) + \Gamma_{\mathbf{v}} q \, dx. \end{aligned} \quad (3.6)$$

As  $\sigma$ ,  $\Gamma_\varphi$  and  $\Gamma_{\mathbf{v}}$  are bounded a.e. in  $Q$  by Lemma 3.1 and (A4), we see that

$$\begin{aligned} \left| \int_{\Omega} \Gamma_\varphi(\mu + \chi\sigma) + \Gamma_{\mathbf{v}} q \, dx \right| &\leq C(1 + \|\mu - \bar{\mu}\|_{L^1} + |\bar{\mu}|_{L^1} + \|q\|_{L^2}) \\ &\leq C(1 + |\bar{\mu}| + \|q\|_{L^2}) + \frac{m_0}{4} \|\nabla \mu\|_{L^2}^2, \end{aligned} \quad (3.7)$$

where we have used the Poincaré inequality (1.5) with  $r = 1$  and Young's inequality. From substituting  $\zeta = 1$  in (2.1b) and using (A3), we find that

$$|\bar{\mu}| \leq C(1 + \|\Psi'(\varphi)\|_{L^1}) \leq C(1 + \|\Psi(\varphi)\|_{L^1}). \quad (3.8)$$

To obtain an estimate of  $\|q\|_{L^2}$ , we look at the pressure system, whose weak formulation is given by (2.1d). Let  $f := (-\Delta_D)^{-1}(q/K)$ , so that

$$\int_{\Omega} K \nabla f \cdot \nabla \phi \, dx = \int_{\Omega} q \phi \, dx \text{ for all } \phi \in H_0^1.$$

Substituting  $\xi = f$  in (2.1d) and  $\phi = q$  in the above leads to

$$\begin{aligned} \|q\|_{L^2}^2 &= \int_{\Omega} K \nabla q \cdot \nabla f \, dx = \int_{\Omega} \Gamma_{\mathbf{v}} f - K \varphi \nabla(\mu + \chi\sigma) \cdot \nabla f \, dx \\ &\leq \|\Gamma_{\mathbf{v}}\|_{L^2} \|f\|_{L^2} + K \|\varphi \nabla(\mu + \chi\sigma)\|_{L^{\frac{6}{5}}} \|\nabla f\|_{L^6} \\ &\leq C(1 + \|\varphi\|_{L^3} \|\nabla(\mu + \chi\sigma)\|_{L^2}) \|f\|_{H^2}. \end{aligned}$$

Using the elliptic regularity estimate  $\|f\|_{H^2} \leq C\|q\|_{L^2}$ , we find that

$$\|q\|_{L^2} \leq C(1 + \|\varphi\|_{H^1} \|\nabla(\mu + \chi\sigma)\|_{L^2}). \quad (3.9)$$

Hence, for the right-hand side of (3.6) we use (3.7), (3.8), (3.9), (A3), the Poincaré inequality (1.5) for  $r = 1$ , and Young's inequality to obtain

$$\begin{aligned} |\text{RHS}| &\leq \frac{3m_0}{4} \|\nabla \mu\|_{L^2}^2 + C \|\nabla \sigma\|_{L^2}^2 + C(1 + \|\Psi(\varphi)\|_{L^1}) + C \|\varphi\|_{H^1}^2 \\ &\leq \frac{3m_0}{4} \|\nabla \mu\|_{L^2}^2 + C(1 + \|\nabla \varphi\|_{L^2}^2 + \|\Psi(\varphi)\|_{L^1} + \|\nabla \sigma\|_{L^2}^2). \end{aligned}$$

Substituting into (3.6) leads to

$$\begin{aligned} \frac{d}{dt} (\|\Psi(\varphi)\|_{L^1} + \|\nabla \varphi\|_{L^2}^2) &- C(\|\Psi(\varphi)\|_{L^1} + \|\nabla \varphi\|_{L^2}^2) \\ &+ \|\nabla \mu\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2 \leq C(1 + \|\nabla \sigma\|_{L^2}^2). \end{aligned}$$

By (A3), (A1) and the Sobolev embedding  $H^1 \subset L^6$ , it holds that  $\Psi(\varphi_0) \in L^1$ . Hence, by an application of Gronwall's inequality, and using the fact that  $\nabla \sigma \in L^2(Q)$ , we obtain

$$\sup_{t \in (0, T]} (\|\Psi(\varphi(t))\|_{L^1} + \|\nabla \varphi(t)\|_{L^2}^2) + \|\nabla \mu\|_{L^2(Q)}^2 + \|\mathbf{v}\|_{L^2(Q)}^2 \leq C.$$

Then, using (3.8) and (A3) and the Poincaré inequality for  $\varphi$  and  $\mu$  yields

$$\sup_{t \in (0, T]} (\|\Psi(\varphi(t))\|_{L^1} + \|\varphi(t)\|_{H^1}^2) + \|\mu\|_{L^2(0, T; H^1)}^2 + \|\mathbf{v}\|_{L^2(Q)}^2 \leq C. \quad (3.10)$$

Next, looking at (2.1b) as an elliptic equation for  $\varphi$ , and using that the potential  $\Psi$  has polynomial growth of order less than 6, we employ the bootstrapping argument in [12, §3.3] and in [13, §4.2] to deduce that

$$\|\Psi'(\varphi)\|_{L^2(0,T;H^1)} + \|\varphi\|_{L^2(0,T;H^3)} \leq C. \quad (3.11)$$

Then, substituting  $\xi = q$  in (2.1d) and the Poincaré inequality (1.7) gives

$$\begin{aligned} K\|\nabla q\|_{L^2}^2 &\leq \|\Gamma_{\mathbf{v}}\|_{L^2}\|q\|_{L^2} + K\|\varphi\nabla(\mu + \chi\sigma)\|_{L^2}\|\nabla q\|_{L^2} \\ &\leq C + \frac{K}{2}\|\nabla q\|_{L^2}^2 + C\|\varphi\|_{L^\infty}^2\|\nabla(\mu + \chi\sigma)\|_{L^2}^2. \end{aligned}$$

By the Gagliardo–Nirenberg inequality (1.8), we have  $\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{H^3}^{\frac{1}{4}}\|\varphi\|_{L^6}^{\frac{3}{4}}$  for three dimensions, and thus we obtain

$$\begin{aligned} \int_0^T \|q\|_{H^1}^{\frac{8}{5}} dt &\leq C \left( 1 + \|\varphi\|_{L^\infty(0,T;H^1)}^{\frac{6}{5}} \int_0^T \|\varphi\|_{H^3}^{\frac{2}{5}} \|\nabla(\mu + \chi\sigma)\|_{L^2}^{\frac{8}{5}} dt \right) \\ &\leq C \left( 1 + \|\varphi\|_{L^2(0,T;H^3)}^{\frac{2}{5}} \|\nabla(\mu + \chi\sigma)\|_{L^2(Q)}^{\frac{8}{5}} \right) \leq C. \end{aligned} \quad (3.12)$$

Lastly, we see that for any  $\zeta \in L^{\frac{8}{3}}(0,T;H^1)$ ,

$$\begin{aligned} \left| \int_Q \varphi \mathbf{v} \cdot \nabla \zeta \right| &\leq \int_0^T \|\varphi\|_{L^\infty} \|\mathbf{v}\|_{L^2} \|\nabla \zeta\|_{L^2} dt \\ &\leq C \|\varphi\|_{L^\infty(0,T;H^1)}^{\frac{3}{4}} \|\mathbf{v}\|_{L^2(Q)} \|\varphi\|_{L^2(0,T;H^3)}^{\frac{1}{4}} \|\zeta\|_{L^{\frac{8}{3}}(0,T;H^1)} \leq C \|\zeta\|_{L^{\frac{8}{3}}(0,T;H^1)}, \end{aligned} \quad (3.13)$$

and so from (2.1a), we obtain

$$\|\partial_t \varphi\|_{L^{\frac{8}{5}}(0,T;(H^1)^*)} \leq C \left( 1 + \|\nabla \mu\|_{L^2(Q)} + \|\operatorname{div}(\varphi \mathbf{v})\|_{L^{\frac{8}{5}}(0,T;(H^1)^*)} \right) \leq C. \quad (3.14)$$

The a priori estimates (3.10), (3.11), (3.12) and (3.14), together with a Galerkin approximation, similar to the one performed in [13, 19] are sufficient to deduce the existence of a weak solution quadruple  $(\varphi, \mu, \mathbf{v}, q)$  to (3.2) with the regularities stated in Lemma 3.2 which satisfies (2.1a), (2.1b), (2.1d), and (2.1e) (with  $\sigma$  replaced by  $\mathcal{L}(\phi)$ ) for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\mathbf{y} \in L^2$ ,  $\xi \in H_0^1$ . We omit the details of the Galerkin procedure and refer the reader to [13] for the details in passing to the limit. Furthermore, the estimate (3.5) is obtained by passing to the limit in the a priori estimates (3.10), (3.11), (3.12) and (3.14) for the Galerkin approximation and using weak/weak\* lower semi-continuity of the norms.

### 3.3 Schauder's fixed point argument

Using the compact embedding  $L^2(0,T;H^1) \cap W^{1,\frac{8}{5}}(0,T;(H^1)^*) \subset\subset L^2(Q)$  from [22, §8, Cor. 4], and by Lemma 3.1 and Lemma 3.2, we can define a compact mapping

$$L^2(Q) \ni \phi \mapsto \mathcal{M}(\phi) := \varphi \in L^2(Q),$$

where  $\varphi$  is the first component of the weak solution to (3.2). To apply Schauder's fixed point theorem and deduce the existence of a fixed point of the mapping  $\mathcal{M}$ , we need to show that there exists a constant  $M$  such that

$$\|\phi\|_{L^2(Q)} \leq M \text{ for all } \phi \in L^2(Q) \text{ and for all } \lambda \in [0, 1] \text{ satisfying } \phi = \lambda \mathcal{M}(\phi).$$

The problem  $\phi = \lambda \mathcal{M}(\phi) = \lambda \varphi$  is equivalent to (1.1a), (1.1b), (1.1c), (1.1e), (1.1f), (1.2a) and

$$0 = \Delta \sigma - h(\lambda \varphi) \sigma \text{ in } \Omega.$$

As  $h, \sigma$  are bounded by 0 and 1 a.e. in  $Q$ , we can choose  $M$  to be the constant  $C$  in (3.5), which does not depend on  $\varphi$  and  $\lambda \in [0, 1]$ . Thus, Schauder's fixed point theorem yields the existence of a weak solution quintuplet  $(\varphi, \mu, \sigma, \mathbf{v}, q)$  to (1.1), (1.2a) in the sense of Definition 2.1.

### 3.4 Robin boundary conditions

We now define the auxiliary problem for (1.1), (1.2b):

$$\operatorname{div} \mathbf{v} = \Gamma_{\mathbf{v}}(\varphi, \mathcal{L}(\phi)) \quad \text{in } Q, \quad (3.15a)$$

$$\mathbf{v} = -K(\nabla p - (\mu + \chi \mathcal{L}(\phi)) \nabla \varphi) \quad \text{in } Q, \quad (3.15b)$$

$$\partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) = \operatorname{div}(m(\varphi) \nabla \mu) + \Gamma_{\varphi}(\varphi, \mathcal{L}(\phi)) \quad \text{in } Q, \quad (3.15c)$$

$$\mu = A\Psi'(\varphi) - B\Delta \varphi - \chi \mathcal{L}(\phi) \quad \text{in } Q, \quad (3.15d)$$

$$\partial_{\nu} \varphi = 0, \quad \mu = 0, \quad K \partial_{\nu} p = a(g - p) \text{ on } \Sigma, \quad (3.15e)$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega. \quad (3.15f)$$

The existence of a weak solution quintuple  $(\varphi, \mu, \sigma, \mathbf{v}, p)$  to (1.1), (1.2b) in the sense of Definition 2.2 can be established with Schauder's fixed point theorem, as done previously in Section 3.3. Hence, it suffices to establish the existence of a weak solution quadruple  $(\varphi, \mu, \mathbf{v}, p)$  to (3.15) analogous to Lemma 3.2. Let us state the existence result to (3.15).

**Lemma 3.3.** *Under Assumption 2.1, for any  $\phi \in L^2(Q)$  there exists a quadruple  $(\varphi, \mu, \mathbf{v}, p)$  such that*

$$\begin{aligned} \varphi &\in L^{\infty}(0, T; H^1) \cap L^2(0, T; H^3) \cap W^{1, \frac{8}{5}}(0, T; H^{-1}), \\ \mathbf{v} &\in L^2(Q), \quad p \in L^{\frac{8}{5}}(0, T; H^1) \cap L^2(\Sigma), \quad \mu \in L^2(0, T; H_0^1), \end{aligned}$$

which satisfies (2.1b), (2.2a), (2.2b), (2.2c) (with  $\sigma$  replaced by  $\mathcal{L}(\phi)$ ) for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\mathbf{y} \in L^2$ ,  $\xi \in H_0^1$ . Furthermore, there exists a positive constant  $C$ , not depending on  $(\varphi, \mu, \mathbf{v}, p)$  and  $\phi$  such that

$$\begin{aligned} &\|\Psi(\varphi)\|_{L^{\infty}(0, T; L^1)} + \|\Psi'(\varphi)\|_{L^2(0, T; H^1)} + \|\varphi\|_{L^{\infty}(0, T; H^1) \cap L^2(0, T; H^3)} + \|\mathbf{v}\|_{L^2(Q)} \\ &+ \|\mu\|_{L^2(0, T; H^1)} + \|p\|_{L^{\frac{8}{5}}(0, T; H^1) \cap L^2(\Sigma)} + \|\partial_t \varphi\|_{L^{\frac{8}{5}}(0, T; H^{-1})} \leq C. \end{aligned} \quad (3.16)$$

Once again we will only derive the a priori estimates and omit the details of the Galerkin approximation. In the following,  $C$  denotes a positive constant not depending on  $(\varphi, \mu, \mathbf{v}, p)$  or  $\phi$ , which may vary from line to line. We write  $\mathcal{L}(\phi)$  as  $\sigma$  in (3.15), substituting  $\zeta = \partial_t \varphi$  in (2.1b),  $\xi = \mu + \chi(\sigma - 1)$  in (2.2a),  $\mathbf{y} = K^{-1} \mathbf{v}$  in (2.2c), and summing leads to

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} A\Psi(\varphi) + \frac{B}{2} |\nabla \varphi|^2 - \chi \varphi \, dx + \int_{\Omega} m(\varphi) |\nabla \mu|^2 + \frac{1}{K} |\mathbf{v}|^2 \, dx + a \|p\|_{L^2(\Gamma)}^2 \\ &= \int_{\Omega} -\chi m(\varphi) \nabla \mu \cdot \nabla \sigma + \Gamma_{\varphi}(\mu + \chi(\sigma - 1)) \, dx \\ &+ \int_{\Omega} p \Gamma_{\mathbf{v}} + \varphi \mathbf{v} \cdot \nabla(\mu + \chi(\sigma - 1)) + (\mu + \chi \sigma) \nabla \varphi \cdot \mathbf{v} \, dx + \int_{\Gamma} a g p \, d\Gamma. \end{aligned} \quad (3.17)$$

Using that  $(\mu + \chi(\sigma - 1)) = 0$  on  $\Gamma$  and the product rule, we have

$$\begin{aligned} & \int_{\Omega} \varphi \mathbf{v} \cdot \nabla(\mu + \chi(\sigma - 1)) + (\mu + \chi\sigma) \nabla \varphi \cdot \mathbf{v} \, dx \\ &= - \int_{\Omega} \Gamma_{\mathbf{v}} \varphi (\mu + \chi(\sigma - 1)) - \chi \mathbf{v} \cdot \nabla \varphi \, dx. \end{aligned}$$

Thus, we obtain the following identity from integrating (3.17) in time

$$\begin{aligned} & \int_{\Omega} \left( A\Psi(\varphi) + \frac{B}{2} |\nabla \varphi|^2 - \chi \varphi \right) (t) \, dx + \int_{\Omega_t} \left( m(\varphi) |\nabla \mu|^2 + \frac{1}{K} |\mathbf{v}|^2 \right) + \int_{\Gamma_t} a |p|^2 \\ &= \int_{\Omega_t} (-\chi m(\varphi) \nabla \mu \cdot \nabla \sigma - \chi \nabla \varphi \cdot \mathbf{v}) + \int_{\Gamma_t} agp \\ &+ \int_{\Omega_t} (\Gamma_{\mathbf{v}}(p - \varphi(\mu + \chi(\sigma - 1))) + \Gamma_{\varphi}(\mu + \chi(\sigma - 1))) \\ &+ \int_{\Omega} \left( A\Psi(\varphi_0) + \frac{B}{2} |\nabla \varphi_0|^2 - \chi \varphi_0 \right) \, dx =: I_1 + I_2 + I_3. \end{aligned} \tag{3.18}$$

Note that by (A3), the third term  $I_3$  on the right-hand side of (3.18) is bounded, and by Young's inequality

$$\left| \int_{\Omega} \chi \varphi \, dx \right| \leq \chi |\Omega|^{\frac{1}{2}} \|\varphi\|_{L^2} \leq \frac{A}{2C_1} \|\varphi\|_{L^2}^2 + C \leq \frac{A}{2} \|\Psi(\varphi)\|_{L^1} + C,$$

which implies that

$$\int_{\Omega} \left( A\Psi(\varphi) + \frac{B}{2} |\nabla \varphi|^2 - \chi \varphi \right) (t) \, dx \geq \frac{A}{2} \|\Psi(\varphi(t))\|_{L^1} + \frac{B}{2} \|\nabla \varphi(t)\|_{L^2}^2 - C.$$

Next, for  $I_1$ , we have

$$\begin{aligned} |I_1| &\leq \frac{m_0}{4} \|\nabla \mu\|_{L^2(Q)}^2 + \frac{1}{2K} \|\mathbf{v}\|_{L^2(Q)}^2 + \frac{a}{2} \|p\|_{L^2(\Sigma)}^2 \\ &+ C \left( \|\nabla \sigma\|_{L^2(Q)}^2 + \|\nabla \varphi\|_{L^2(Q)}^2 + \|g\|_{L^2(\Sigma)}^2 \right). \end{aligned}$$

It remains to estimate  $I_2$ , and we first obtain an estimate on  $\|p\|_{L^2}$  by looking at the pressure system, whose weak formulation is given by (2.2b). Let  $f := (-\Delta_R)^{-1}(p/K, a/K, 0)$ , so that

$$\int_{\Omega} K \nabla f \cdot \nabla \phi \, dx + \int_{\Gamma} a f \phi \, d\Gamma = \int_{\Omega} p \phi \, dx \text{ for all } \phi \in H^1.$$

Substituting  $\zeta = f$  in (2.2b) and  $\phi = p$  in the above leads to

$$\begin{aligned} \|p\|_{L^2}^2 &= \int_{\Omega} \Gamma_{\mathbf{v}} f + K(\mu + \chi\sigma) \nabla \varphi \cdot \nabla f \, dx + \int_{\Gamma} agf \, d\Gamma \\ &\leq \|\Gamma_{\mathbf{v}}\|_{L^2} \|f\|_{L^2} + K \|(\mu + \chi\sigma) \nabla \varphi\|_{L^{\frac{6}{5}}} \|\nabla f\|_{L^6} + a \|g\|_{L^2(\Gamma)} \|f\|_{L^2(\Gamma)} \\ &\leq C \left( 1 + \|g\|_{L^2(\Gamma)} + \|(\mu + \chi\sigma) \nabla \varphi\|_{L^{\frac{6}{5}}} \right) \|f\|_{H^2}. \end{aligned} \tag{3.19}$$

Using the elliptic regularity estimate  $\|f\|_{H^2} \leq C \|p\|_{L^2}$ , we obtain, analogous to (3.9),

$$\begin{aligned} \|p\|_{L^2} &\leq C \left( 1 + \|g\|_{L^2(\Gamma)} + \|(\mu + \chi\sigma) \nabla \varphi\|_{L^{\frac{6}{5}}} \right) \\ &\leq C \left( 1 + \|g\|_{L^2(\Gamma)} + \|\mu + \chi\sigma\|_{L^6} \|\nabla \varphi\|_{L^{\frac{3}{2}}} \right) \\ &\leq C \left( 1 + \|g\|_{L^2(\Gamma)} + (\|\nabla \mu\|_{L^2} + \chi \|\sigma\|_{H^1}) \|\nabla \varphi\|_{L^{\frac{3}{2}}} \right), \end{aligned} \tag{3.20}$$

where we have used the Poincaré inequality (1.7) and the Sobolev embedding  $H^1 \subset L^6$ . Using the boundedness of  $\sigma$ ,  $\Gamma_{\mathbf{v}}$  and  $\Gamma_{\varphi}$ , (A3) and  $\|p\|_{L^1(Q)} \leq C\|p\|_{L^1(0,T;L^2)}$ , we see that

$$\begin{aligned} |I_2| &\leq C \left( 1 + \|p\|_{L^1(Q)} + \|\varphi\|_{L^2(Q)} \|\mu\|_{L^2(Q)} + \|\varphi\|_{L^1(Q)} + \|\mu\|_{L^1(Q)} \right) \\ &\leq C \left( 1 + \|g\|_{L^2(\Sigma)} + \|\sigma\|_{L^2(0,T;H^1)}^2 + \|\nabla\varphi\|_{L^2(Q)}^2 + \|\varphi\|_{L^2(Q)}^2 \right) + \frac{m_0}{4} \|\nabla\mu\|_{L^2(Q)}^2 \\ &\leq C \left( 1 + \|\Psi(\varphi)\|_{L^1(Q)} + \|\nabla\varphi\|_{L^2(Q)}^2 + \|g\|_{L^2(\Sigma)}^2 + \|\sigma\|_{L^2(0,T;H^1)}^2 \right) + \frac{m_0}{4} \|\nabla\mu\|_{L^2(Q)}^2. \end{aligned}$$

Thus, we obtain from (3.18) the inequality

$$\begin{aligned} &(\|\Psi(\varphi(t))\|_{L^1} + \|\nabla\varphi(t)\|_{L^2}^2) + \|\nabla\mu\|_{L^2(Q)}^2 + \|\mathbf{v}\|_{L^2(Q)}^2 + \|p\|_{L^2(\Sigma)}^2 \\ &\leq C \left( 1 + \|g\|_{L^2(\Sigma)}^2 + \|\sigma\|_{L^2(0,T;H^1)}^2 + \|\Psi(\varphi)\|_{L^1(Q)} + \|\nabla\varphi\|_{L^2(Q)}^2 \right) \text{ for } t \in (0, T]. \end{aligned}$$

Applying the integral version of Gronwall's inequality, see for example [14, Lem. 3.1], we obtain

$$\sup_{t \in (0, T]} (\|\Psi(\varphi(t))\|_{L^1} + \|\nabla\varphi(t)\|_{L^2}^2) + \|\nabla\mu\|_{L^2(Q)}^2 + \|\mathbf{v}\|_{L^2(Q)}^2 + \|p\|_{L^2(\Sigma)}^2 \leq C. \quad (3.21)$$

Then, using (A3) and the Poincaré inequality for  $\mu$ , this yields

$$\sup_{t \in (0, T]} (\|\Psi(\varphi(t))\|_{L^1} + \|\varphi(t)\|_{H^1}^2) + \|\mu\|_{L^2(0,T;H^1)}^2 + \|\mathbf{v}\|_{L^2(Q)}^2 + \|p\|_{L^2(\Sigma)}^2 \leq C. \quad (3.22)$$

Analogous to the Dirichlet case, a bootstrapping argument akin to [12, §3.3] and [13, §4.2] leads to the estimate

$$\|\Psi'(\varphi)\|_{L^2(0,T;H^1)} + \|\varphi\|_{L^2(0,T;H^3)} \leq C. \quad (3.23)$$

Then, from (2.2b) and the Poincaré inequality (1.6), it holds that

$$\begin{aligned} K\|\nabla p\|_{L^2}^2 + \frac{a}{2}\|p\|_{L^2(\Gamma)}^2 &\leq \|\Gamma_{\mathbf{v}}\|_{L^2}\|p\|_{L^2} + K\|(\mu + \chi\sigma)\nabla\varphi\|_{L^2}\|\nabla p\|_{L^2} + \frac{a}{2}\|g\|_{L^2(\Gamma)}^2 \\ &\leq C \left( 1 + \|g\|_{L^2(\Gamma)}^2 \right) + \frac{K}{2}\|\nabla p\|_{L^2}^2 + \frac{a}{4}\|p\|_{L^2(\Gamma)}^2 + K\|(\mu + \chi\sigma)\nabla\varphi\|_{L^2}^2, \end{aligned}$$

which implies that

$$\|p\|_{H^1} \leq C \left( 1 + \|g\|_{L^2(\Gamma)} + \|(\mu + \chi\sigma)\nabla\varphi\|_{L^2} \right). \quad (3.24)$$

By the Gagliardo–Nirenberg inequality (1.8), we see that

$$\|\nabla\varphi\|_{L^3} \leq C\|\varphi\|_{H^3}^{\frac{1}{4}}\|\varphi\|_{L^6}^{\frac{3}{4}} \quad (3.25)$$

for three dimensions, and thus  $(\mu + \chi\sigma)\nabla\varphi \in L^{\frac{8}{3}}(0, T; L^2)$ . From (3.24) this implies that

$$\|p\|_{L^{\frac{8}{5}}(0,T;H^1)} \leq C. \quad (3.26)$$

Analogous to (3.13), for  $\xi \in L^{\frac{8}{3}}(0, T; H_0^1)$ , using that  $\varphi \in L^\infty(0, T; H^1) \cap L^2(0, T; H^3)$  and  $\mathbf{v} \in L^2(Q)$ , it holds that

$$\left| \int_Q \varphi \mathbf{v} \cdot \nabla \xi \right| \leq C \|\xi\|_{L^{\frac{8}{3}}(0,T;H_0^1)},$$

which in turn implies that

$$\|\partial_t \varphi\|_{L^{\frac{8}{5}}(0,T;H^{-1})} \leq C \quad (3.27)$$

by the inspection of (2.2a). The a priori estimates (3.22), (3.23), (3.26) and (3.27), together with a Galerkin approximation are sufficient to deduce the existence of a weak solution quadruple  $(\varphi, \mu, \mathbf{v}, p)$  to (3.15) with the regularities stated in Lemma 3.3 which satisfies (2.1b), (2.2a), (2.2b) and (2.2c) (with  $\sigma$  replaced by  $\mathcal{L}(\phi)$ ) for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\mathbf{y} \in L^2$ ,  $\xi \in H_0^1$ . The estimate (3.16) follows from passing to the limit in the a priori estimates (3.22), (3.23), (3.26) and (3.27) for the Galerkin approximation and using weak/weak\* lower semi-continuity of the norms. Then, a similar Schauder's fixed point argument to Section 3.3 can be applied by choosing the constant  $M$  to be the constant  $C$  in (3.16).

**Remark 3.1.** *The necessity of a Dirichlet condition for  $\mu$  in (3.15) is due to the fact that we cannot control  $\|\mu \nabla \varphi\|_{L^{\frac{6}{5}}}$  in (3.20) simply with the left-hand side of (3.18) if we assume  $\partial_\nu \mu = 0$  on  $\Sigma$ . One could consider the splitting*

$$\begin{aligned} \|\mu \nabla \varphi\|_{L^{\frac{6}{5}}} &\leq \|(\mu - \bar{\mu}) \nabla \varphi\|_{L^{\frac{6}{5}}} + |\bar{\mu}| \|\nabla \varphi\|_{L^{\frac{6}{5}}} \leq \|\mu - \bar{\mu}\|_{L^6} \|\nabla \varphi\|_{L^{\frac{3}{2}}} + |\bar{\mu}| \|\nabla \varphi\|_{L^{\frac{6}{5}}} \\ &\leq C \|\nabla \mu\|_{L^2} \|\nabla \varphi\|_{L^{\frac{3}{2}}} + C (1 + \|\Psi'(\varphi)\|_{L^1}) \|\nabla \varphi\|_{L^{\frac{6}{5}}}, \end{aligned}$$

and in order to control the second term, it is desirable to have an estimate of the form

$$\|\Psi'(\varphi)\|_{L^1}^2 \leq C (1 + \|\Psi(\varphi)\|_{L^1}).$$

This leads to the situation encountered in [14] and restricts  $\Psi$  to have quadratic growth. Furthermore, the ansatz in [13, 19] is to consider the splitting

$$\begin{aligned} \left| \int_{\Omega} \Gamma_{\mathbf{v}}(p - \mu \varphi) \, dx \right| &= \left| \int_{\Omega} \Gamma_{\mathbf{v}}(p - \bar{\mu} \varphi) + \Gamma_{\mathbf{v}}(\bar{\mu} - \mu) \varphi \, dx \right| \\ &\leq \left| \int_{\Omega} \Gamma_{\mathbf{v}}(p - \bar{\mu} \varphi) \, dx \right| + C \|\nabla \mu\|_{L^2} \|\varphi\|_{L^2}. \end{aligned}$$

If  $p$  satisfies the Darcy law (1.2b) with the boundary condition  $\partial_\nu p = 0$  on  $\Sigma$ , and if  $\Gamma_{\mathbf{v}}$  has zero mean, then we can write

$$p = (-\Delta_N)^{-1} (\Gamma_{\mathbf{v}}/K - \operatorname{div}((\mu - \bar{\mu} + \chi \sigma) \nabla \varphi) - \bar{\mu} \operatorname{div}(\nabla(\varphi - \bar{\varphi}))),$$

where for  $f \in L^2$  with  $\bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f \, dx = 0$ , we denote  $u := (-\Delta_N)^{-1}(f) \in H^1$  as the unique weak solution to

$$-\Delta u = f \text{ in } \Omega, \quad \partial_\nu u = 0 \text{ on } \Gamma \text{ with } \bar{u} = 0.$$

A short calculation shows that

$$-(-\Delta_N)^{-1}(\operatorname{div}(\bar{\mu} \nabla(\varphi - \bar{\varphi}))) = \bar{\mu}(\varphi - \bar{\varphi}),$$

and so

$$\int_{\Omega} \Gamma_{\mathbf{v}}(p - \bar{\mu} \varphi) \, dx = \int_{\Omega} \Gamma_{\mathbf{v}}((- \Delta_N)^{-1}(\Gamma_{\mathbf{v}}/K - \operatorname{div}((\mu - \bar{\mu} + \chi \sigma) \nabla \varphi))) - \Gamma_{\mathbf{v}} \bar{\mu} \bar{\varphi} \, dx.$$

In [13, 19],  $\Gamma_{\mathbf{v}}$  has zero mean, and so the last term vanishes, but this is not the case in our present setting, and thus the approach of [13, 19] seems not to give any advantage in deriving a priori estimates.

### 3.5 Quadratic potentials

In this section, let us state an analogous result to Lemma 3.3 for the auxiliary problem (3.15), but now we consider

$$\partial_{\nu}\varphi = \partial_{\nu}\mu = 0, \quad K\partial_{\nu}p = a(g - p) \text{ on } \Sigma, \quad (3.28)$$

and (2.4) instead of (A3). The assertion is formulated as follows.

**Lemma 3.4.** *Under Assumption 2.1 (with (2.4) instead of (A3)), for any  $\phi \in L^2(Q)$  there exists a quadruple  $(\varphi, \mu, \mathbf{v}, p)$  such that*

$$\begin{aligned} \varphi &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^3) \cap W^{1, \frac{8}{5}}(0, T; (H^1)^*), \\ \mathbf{v} &\in L^2(Q), \quad p \in L^{\frac{8}{5}}(0, T; H^1) \cap L^2(\Sigma), \quad \mu \in L^2(0, T; H^1), \end{aligned}$$

which satisfies (2.1b), (2.2b), (2.2c), (2.3) (with  $\sigma$  replaced by  $\mathcal{L}(\phi)$ ) for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\mathbf{y} \in L^2$ . Furthermore, there exists a positive constant  $C$ , not depending on  $(\varphi, \mu, \mathbf{v}, p)$  and  $\phi$  such that

$$\begin{aligned} &\|\Psi(\varphi)\|_{L^\infty(0, T; L^1)} + \|\Psi'(\varphi)\|_{L^2(0, T; H^1)} + \|\varphi\|_{L^\infty(0, T; H^1) \cap L^2(0, T; H^3)} + \|\mathbf{v}\|_{L^2(Q)} \\ &+ \|\mu\|_{L^2(0, T; H^1)} + \|p\|_{L^{\frac{8}{5}}(0, T; H^1) \cap L^2(\Sigma)} + \|\partial_t \varphi\|_{L^{\frac{8}{5}}(0, T; (H^1)^*)} \leq C. \end{aligned} \quad (3.29)$$

Once again we will only derive the a priori estimates and omit the details of the Galerkin approximation. Substituting  $\zeta = \mu + \chi\sigma$  into (2.3), and upon adding with the equalities obtained from substituting  $\zeta = \partial_t \varphi$  in (2.1b) and  $\mathbf{y} = K^{-1}\mathbf{v}$  in (2.2c) we have

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} A\Psi(\varphi) + \frac{B}{2} |\nabla \varphi|^2 \, dx + \int_{\Omega} m(\varphi) |\nabla \mu|^2 + \frac{1}{K} |\mathbf{v}|^2 \, dx + \int_{\Gamma} a |p|^2 \, d\Gamma \\ &= \int_{\Omega} -\chi m(\varphi) \nabla \mu \cdot \nabla \sigma + \Gamma_{\varphi}(\mu + \chi\sigma) + \Gamma_{\mathbf{v}}(p - \varphi(\mu + \chi\sigma)) \, dx + \int_{\Gamma} agp \, d\Gamma. \end{aligned} \quad (3.30)$$

The first term and the boundary term on the right-hand side can be handled using Hölder's inequality and Young's inequality. From the computations in Section 3.4 and the discussion in Remark 3.1, we obtain

$$\begin{aligned} \|p\|_{L^2} &\leq C \left( 1 + \|g\|_{L^2(\Gamma)} + \|(\mu + \chi\sigma) \nabla \varphi\|_{L^{\frac{6}{5}}} \right) \\ &\leq C \left( 1 + \|g\|_{L^2(\Gamma)} + \|\nabla \mu\|_{L^2} \|\nabla \varphi\|_{L^{\frac{3}{2}}} + (|\bar{\mu}| + \chi) \|\nabla \varphi\|_{L^{\frac{6}{5}}} \right). \end{aligned}$$

Substituting  $\zeta = 1$  in (2.1b), we can estimate the mean of  $\mu$  by

$$|\bar{\mu}| \leq C (\chi \|\sigma\|_{L^1} + \|\Psi'(\varphi)\|_{L^1}) \leq C (1 + \|\Psi'(\varphi)\|_{L^1}), \quad (3.31)$$

and so by Young's inequality and the boundedness of  $\Gamma_{\mathbf{v}}$ , we see that

$$\begin{aligned} |X| &:= \left| \int_{\Omega} \Gamma_{\mathbf{v}}(p - \varphi(\mu - \bar{\mu}) - \varphi(\bar{\mu} + \chi\sigma)) \, dx \right| \\ &\leq C (\|p\|_{L^2} + \|\varphi\|_{L^2} \|\nabla \mu\|_{L^2} + (1 + \|\Psi'(\varphi)\|_{L^1}) \|\varphi\|_{L^1}) \\ &\leq C (1 + \|g\|_{L^2(\Gamma)} + (1 + \|\nabla \mu\|_{L^2} + \|\Psi'(\varphi)\|_{L^1}) \|\varphi\|_{H^1}) \\ &\leq \frac{m_0}{4} \|\nabla \mu\|_{L^2}^2 + C (1 + \|g\|_{L^2(\Gamma)} + \|\Psi'(\varphi)\|_{L^1}^2 + \|\varphi\|_{L^2}^2 + \|\nabla \varphi\|_{L^2}^2). \end{aligned}$$



Using that  $\Psi$  has quadratic growth, we can find positive constants  $C_4, C_5$  such that

$$|\Psi'(s)| \leq C_4 |s| + C_5 \quad \forall s \in \mathbb{R},$$

and so by (2.4)

$$\|\Psi'(\varphi)\|_{L^1}^2 \leq C (1 + \|\varphi\|_{L^2}^2) \leq C (1 + \|\Psi(\varphi)\|_{L^1}). \quad (3.32)$$

This implies that

$$|X| \leq \frac{m_0}{4} \|\nabla \mu\|_{L^2}^2 + C \left( 1 + \|g\|_{L^2(\Gamma)}^2 + \|\Psi(\varphi)\|_{L^1} + \|\nabla \varphi\|_{L^2}^2 \right).$$

In a similar fashion, the second term on the right-hand side of (3.30) can be estimated as

$$\begin{aligned} \left| \int_{\Omega} \Gamma_{\varphi}(\mu - \bar{\mu} + \bar{\mu} + \chi\sigma) \, dx \right| &\leq C (1 + |\bar{\mu}| + \|\nabla \mu\|_{L^2}) \\ &\leq \frac{m_0}{4} \|\nabla \mu\|_{L^2}^2 + C (1 + \|\Psi(\varphi)\|_{L^1}), \end{aligned}$$

and we obtain from (3.30)

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} A\Psi(\varphi) + \frac{B}{2} |\nabla \varphi|^2 \, dx + \frac{m_0}{4} \|\nabla \mu\|_{L^2}^2 + \frac{1}{K} \|\mathbf{v}\|_{L^2}^2 + \frac{a}{2} \|p\|_{L^2(\Gamma)}^2 \\ &\leq C \left( 1 + \|g\|_{L^2(\Gamma)}^2 + \|\Psi(\varphi)\|_{L^1} + \|\nabla \varphi\|_{L^2}^2 \right). \end{aligned} \quad (3.33)$$

Applying Gronwall's inequality leads to (3.21), and the a priori estimate (3.22) follows by applying the Poincaré inequality (1.5), (3.31) and (3.32). The other a priori estimates (3.23), (3.26) follow from a similar argument. For the time derivative  $\partial_t \varphi$ , we note that  $\nabla \varphi \cdot \mathbf{v} \in L^{\frac{8}{5}}(0, T; (H^1)^*)$  by (3.25), and so from (2.3) it holds that

$$\|\partial_t \varphi\|_{L^{\frac{8}{5}}(0, T; (H^1)^*)} \leq C. \quad (3.34)$$

The a priori estimates (3.22), (3.23), (3.26) and (3.34), together with a Galerkin approximation are sufficient to deduce the existence of a weak solution quadruple  $(\varphi, \mu, \mathbf{v}, p)$  to (3.15) with the boundary conditions (3.28) and the regularities stated in Lemma 3.4 which satisfies (2.1b), (2.2b), (2.2c) and (2.3) (with  $\sigma$  replaced by  $\mathcal{L}(\phi)$ ) for a.e.  $t \in (0, T)$  and all  $\zeta \in H^1$ ,  $\mathbf{y} \in L^2$ . The estimate (3.29) follows from passing to the limit in the a priori estimates (3.22), (3.23), (3.26) and (3.34) for the Galerkin approximation and using weak/weak\* lower semi-continuity of the norms. Then, a similar Schauder's fixed point argument to Section 3.3 can be applied by choosing the constant  $M$  to be the constant  $C$  in (3.29).

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